

# A Decomposition of Irreversible Diffusion Processes Without Detailed Balance

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## Abstract

As an generalization of deterministic, nonlinear conservative systems, a notion of *canonical conservative dynamics* with respect to a positive, differentiable stationary density  $\rho(x)$  is introduced:  $\dot{x} = j(x)$  in which  $\nabla \cdot (\rho(x)j(x)) = 0$ . Such systems have a conserved “free energy function”  $F[u] = \int u(x, t) \ln(u(x, t)/\rho(x)) dx$  in the phase space with density flow  $u(x, t)$  satisfying  $u_t = -\nabla \cdot (ju)$ . Any general stochastic diffusion process without detailed balance, in terms of its Fokker-Planck equation, can then be decomposed into a reversible diffusion process with detailed balance and a canonical conserved system. This decomposition can be rigorously established in a functional space with inner product defined as  $\langle \phi, \psi \rangle = \int \rho^{-1}(x) \phi(x) \psi(x) dx$ . Furthermore, a law for balancing  $F[u]$  can be obtained: The non-positive  $dF[u(x, t)]/dt = Q_{hk}(t) - e_p(t)$  where the “source”  $Q_{hk}(t) \geq 0$  and the “sink”  $e_p(t) \geq 0$  are known as energy pumping and entropy production, respectively. A reversible diffusion has  $Q_{hk} = 0$ . For a linear (Ornstein-Uhlenbeck) diffusion process, our decomposition is equivalent to the previous approaches developed by R. Graham and P. Ao, as well as the theory of large deviations.

## 1 Introduction

With the recent development of stochastic thermodynamics in terms of mesoscopic entropy production [1, 2, 3], free energy dissipation [4, 5], work equalities and fluctuation theorems [6], and the mathematical theory of nonequilibrium steady state [7, 8, 9], there is a revitalized interest in stochastic nonlinear dynamics [10], particularly those without detailed balance [11], also known as irreversible Markov processes.

For a general discrete state Markov process, either with discrete or continuous time, a decomposition theorem is known [7, 12]. In a nutshell, the transition probability matrix of a Markov process, with respect to its stationary distribution, can be decomposed in terms of a symmetric and an anti-symmetric parts. The latter part can be further decomposed into many pure rotations among the discrete states. The notion of cycle kinetics arises in this analysis [13, 14].

For continuous diffusion processes on  $\mathbb{R}^n$  without detailed balance, such a decomposition has never been fully established, even though computations have revealed both stationary density and rotational flux as key determinants of a stationary process [15]. A sophisticated analysis on a compact differentiable manifold also exists [7, 16]. In the physics literature, R. Graham and coworkers have proposed and studied extensively a decomposition of non-gradient vector field in terms of Fokker-Planck equations, via WKB method and a Hamilton-Jacobi equation [17, 18, 19]. But the program was ultimately abandoned due to technical difficulties [20]. In recent years, P. Ao and coworkers have again proposed a related decomposition from a rather different starting point [21, 22, 23, 24, 25]. However, the feasibility of this new approach has been rigorously demonstrated only for a linear system [21] which is nearly equivalent, apart from the normalizability of the invariant density, to analyzing irreversible Ornstein-Uhlenbeck processes [26]. Still, their emphasis on stable as well as unstable fixed points had suggested the possible applicability to nonlinear systems. A full analysis beyond heuristic for general nonlinear diffusion processes without detailed balance still is not available. See [23] and [27] for more, and recent discussions.

The central piece of the Fokker-Planck equation of a diffusion process is a linear, partial differential operator  $\mathcal{L}$  [28, 7]. Assuming the existence of a unique stationary density  $\rho(x) > 0$ ,  $x \in \mathbb{R}^n$  and  $\mathcal{L}(\rho) = 0$ , an inner product in a functional space

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^n} \phi(x) \psi(x) \rho^{-1}(x) dx \quad (1)$$

can be introduced [28, 7]. For diffusion process with detailed balance, this functional analysis approach is reduced to the Sturm-Liouville problem. In the present work, we follow this approach and introduce an operator decomposition in terms of a symmetric  $\mathcal{L}_s$  and an anti-symmetric  $\mathcal{L}_a$ :  $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_a$ . While the partial differential equation (PDE)  $(\partial/\partial t - \mathcal{L}_s)(u) = 0$  representing a reversible diffusion process is a gradient-like system, as expected for a symmetric operator, the PDE  $(\partial/\partial t - \mathcal{L}_a)(u) = 0$  turns out to be a hyperbolic system.

We show that the anti-symmetric system is in fact a generalization of the conservative nonlinear dynamics one usually studies [29] where the  $\rho$  being uniform. If one calls the usual conservative dynamics  $\dot{x} = g(x)$  with  $\nabla \cdot g(x) = 0$  a *microcanonical system*, then the dynamical system defined by  $\mathcal{L}_a$  with  $\dot{x} = j(x)$  in which  $\nabla \cdot (\rho(x)j(x)) = 0$  could be called a *canonical system*. Indeed, for the density function  $u(x, t)$  in the phase space, a microcanonical dynamics has

$$\frac{d}{dt} \left( - \int_{\mathbb{R}^n} u(x, t) \ln u(x, t) dx \right) = 0, \quad (2a)$$

as was known to Boltzmann, while a canonical dynamics has

$$\frac{d}{dt} \left[ \int_{\mathbb{R}^n} u(x, t) \ln \left( \frac{u(x, t)}{\rho(x)} \right) dx \right] = 0. \quad (2b)$$

Eqs. (2a) and (2b) can be used as the definitions for microcanonical and canonical conservative dynamics, respectively.

## 2 Fokker-Planck differential operator decomposition

A general Fokker-Planck equation is characterized by a symmetric  $n \times n$  diffusion matrix  $A(x)$  and a vector field,  $b(x)$ , the drift. In the present work, we shall denote the linear differential operator in a functional space

$$\mathcal{L}(\phi) = \nabla (A(x) \nabla \phi(x)) - \nabla (b(x) \phi(x)), \quad (3)$$

$$\mathcal{L}^*(\phi) = \nabla (A(x) \nabla \phi(x)) + b(x) \nabla \phi(x), \quad (4)$$

and adopt the inner production given in Eq. (1), in which  $\rho(x)$  is the normalized stationary solution to the PDE  $\mathcal{L}(\rho) = 0$ . We assume it is unique, positive and differentiable [28, 7]. Then we have

$$\begin{aligned} \langle \psi, \mathcal{L}(\phi) \rangle &= \int_{\mathbb{R}^n} \rho^{-1} \psi(x) dx \nabla (A(x) \nabla \phi(x)) - \nabla (b(x) \phi(x)) \\ &= \int_{\mathbb{R}^n} \{ \nabla [A(x) \nabla (\rho^{-1} \psi(x))] + b(x) \nabla (\rho^{-1} \psi(x)) \} \phi(x) dx \\ &= \langle \rho \mathcal{L}^* (\rho^{-1} \psi), \phi \rangle. \end{aligned}$$

We now introduce symmetric and anti-symmetric operators

$$\mathcal{L}_s(\phi) = \frac{1}{2} \{ \mathcal{L}(\phi) + \rho \mathcal{L}^* (\rho^{-1} \phi) \}, \quad (5)$$

$$\mathcal{L}_a(\phi) = \frac{1}{2} \{ \mathcal{L}(\phi) - \rho \mathcal{L}^* (\rho^{-1} \phi) \}. \quad (6)$$

Then,

$$\langle \psi, \mathcal{L}_s(\phi) \rangle = \frac{1}{2} \left( \langle \psi, \mathcal{L}(\phi) \rangle + \langle \mathcal{L}(\psi), \phi \rangle \right) = \langle \mathcal{L}_s(\psi), \phi \rangle, \quad (7)$$

$$\langle \psi, \mathcal{L}_a(\phi) \rangle = -\langle \mathcal{L}_a(\psi), \phi \rangle, \quad (8)$$

and  $\mathcal{L}$  has been decomposed into  $(\mathcal{L}_s + \mathcal{L}_a)$ .

### 3 Symmetric reversible diffusion

The partial differential operator

$$\mathcal{L}_s(u) = \nabla \left( A(x) \nabla u(x) - (A(x) \nabla \ln \rho(x)) u(x) \right) \quad (9)$$

is a Fokker-Planck type with diffusion tensor  $A(x)$  and drift  $A(x) \nabla \ln \rho(x)$ . One can immediately see that  $\rho(x)$  is its stationary density. The diffusion process associated with  $\mathcal{L}_s$  has been extensively studied in the past and is well understood.

The corresponding partial differential equation (PDE)

$$\frac{\partial u(x, t)}{\partial t} = \mathcal{L}_s(u(x, t)) \quad (10)$$

is formally a gradient system

$$\frac{\partial u}{\partial t} = \frac{\delta}{\delta u} \frac{\langle \mathcal{L}_s(u), u \rangle}{2}. \quad (11)$$

See [28] for many important properties associated with reversible diffusion processes.

**Potential function, entropy production and stochastic free energy.** The “potential function” in Eq. (11)

$$-\langle \mathcal{L}_s(u), u \rangle = \int_{\mathbb{R}^n} \left[ \nabla \left( \frac{u(x, t)}{\rho(x)} \right) \right]^T A(x) \left[ \nabla \ln \left( \frac{u(x, t)}{\rho(x)} \right) \right] u(x, t) dx. \quad (12)$$

The gradient system in (11) is confined on an affine subspace with  $\int_{\mathbb{R}^n} u(x) dx = \langle \rho(x), u(x) \rangle = 1$ . The PDE (10) has also been shown as a gradient flow generated by a potential function  $F[u]$  on an appropriate Riemann manifold with Wasserstein metric [30, 31]. The  $F[u]$  is known as the “stochastic free energy” for the Markov system [32, 25, 4, 5]:

$$F[u(x, t)] = \int_{\mathbb{R}^n} u(x, t) \ln \left( \frac{u(x, t)}{\rho(x)} \right) dx = \langle \rho u, \ln(\rho^{-1} u) \rangle. \quad (13)$$

Then for  $u(x, t)$  following the self-adjoint PDE (10) [33, 4, 5],

$$\begin{aligned}\frac{dF(t)}{dt} &= \left\langle \rho \frac{\partial u}{\partial t}, \ln(\rho^{-1}u) \right\rangle + \left\langle \rho u, u^{-1} \frac{\partial u}{\partial t} \right\rangle \\ &= \left\langle \mathcal{L}_s(u), \rho \ln(\rho^{-1}u) \right\rangle = -e_p.\end{aligned}\quad (14)$$

The  $e_p \geq 0$  is known as the entropy production rate for the reversible diffusion process [28, 7]. A stationary reversible diffusion, therefore, has  $e_p = 0$ .

**Stochastic differential equation for trajectories.** One can also write a stochastic differential equation for the diffusion process described by Eq. (10). If we follow Ito's notion of stochastic integration, we have

$$dx_i(t) = \rho^{-1}(\vec{x}) \sum_j \left( \frac{\partial}{\partial x_j} A_{ij}(\vec{x}) \rho(\vec{x}) \right) dt + \sum_j \Gamma_{ij}(\vec{x}) dB_j(t), \quad (15)$$

in which matrix  $A(\vec{x}) = \frac{1}{2} \Gamma(\vec{x}) \Gamma^T(\vec{x})$ , and  $B_i(t)$  are standard Brownian motions. On the other hand, if one follows Stratonovich's integration, one has

$$dx_i(t) = \sum_j \left( A_{ij}(\vec{x}) \frac{\partial \ln \rho(\vec{x})}{\partial x_j} + \frac{1}{2} \sum_k \Gamma_{ik}(\vec{x}) \frac{\partial}{\partial x_j} \Gamma_{jk}(\vec{x}) \right) dt + \sum_j \Gamma_{ij}(\vec{x}) dB_j(t), \quad (16)$$

and if one takes the “divergence form” for the integration, as strongly advocated by P. Ao [22, 24], then

$$dx_i(t) = \sum_j \left( A_{ij}(\vec{x}) \frac{\partial}{\partial x_j} \ln \rho(\vec{x}) \right) dt + \sum_j \Gamma_{ij}(\vec{x}) dB_j(t). \quad (17)$$

## 4 Canonical conservative dynamics

On the other hand, the anti-symmetric partial differential operator

$$\mathcal{L}_a(u) = \nabla \left( (A(x) \nabla \ln \rho(x) - b(x)) u(x) \right) \quad (18)$$

$$= (A(x) \nabla \rho(x) - b(x) \rho(x)) \cdot \nabla (\rho^{-1}(x) u(x)). \quad (19)$$

The corresponding PDE

$$\frac{\partial u(x, t)}{\partial t} = \mathcal{L}_a(u(x, t)) \quad (20)$$

is not diffusive but rather it is hyperbolic. It is easy to verify that  $\rho(x)$  is again a stationary density for Eq. (20). Eq. (19) actually is the Liouville equation in the phase space for the

nonlinear ordinary differential equation (ODE)

$$\frac{dx}{dt} = b(x) - A(x) \nabla \ln \rho(x) \equiv j(x). \quad (21)$$

The  $j(x)$  in Eq. (21) satisfies

$$\nabla \cdot (\rho(x) j(x)) = 0. \quad (22)$$

That is,

$$\nabla \cdot j(x) + j(x) \cdot \nabla \ln \rho(x) = 0. \quad (23)$$

We shall call Eq. (21) “canonical conservative system” with respect to stationary density  $\rho(x)$ . Since  $\rho(x)$  is an invariant density to the dynamics in Eq. (20), one can again consider the free energy functional [32, 25, 4, 5]

$$F[u(x, t)] = \int_{\mathbb{R}^n} u(x, t) \ln \left( \frac{u(x, t)}{\rho(x)} \right) dx. \quad (24)$$

For the hyperbolic system, this is a generalization of Boltzmann’s  $H$ -function in which the stationary distribution  $\rho(x) = \text{constant}$  due to “equal probability a priori”. Then one has [34, 32]

$$\begin{aligned} \frac{dF(t)}{dt} &= \int_{\mathbb{R}^n} \frac{\partial u(x, t)}{\partial t} \ln \left( \frac{u(x, t)}{\rho(x)} \right) dx \\ &= \int_{\mathbb{R}^n} j(x) u(x, t) \left[ \frac{\nabla u(x, t)}{u(x, t)} - \frac{\nabla \rho(x)}{\rho(x)} \right] dx \\ &= - \int_{\mathbb{R}^n} (\nabla \cdot j(x) + j(x) \cdot \nabla \ln \rho(x)) u(x, t) dx \\ &= 0. \end{aligned} \quad (25)$$

In the derivation we have used Eqs. (20) and (23) as well as assumed that  $u(x, t) \rightarrow 0$  sufficiently fast when  $\|x\| \rightarrow \infty$ .

The result in Eq. (25) is reminiscent of a Boltzmann’s result before he introduced his *Stosszahlansatz*. His  $H$ -function,  $-\int f(x, t) \ln f(x, t) dx$ , actually is invariant with respect to time if  $f(x, t)$  follows strictly the phase space Liouville equation for a Hamiltonian dynamics. One also notes that the  $L_2$  norm of  $u(x, t)$  is conserved in a canonical conservative dynamics:  $\frac{d}{dt} \|u(x, t)\|^2 = \langle \mathcal{L}_a(u), u \rangle = -\langle u, \mathcal{L}_a(u) \rangle = 0$ . Actually, any functional  $\int_{\mathbb{R}^n} u G(u/\rho) dx$  is a conserved quantity for Eq. (20).

**Fixed points in canonical conservative system.** Eq. (23) indicates that  $\nabla \cdot j(x) = 0$  at the fixed point of vector field  $j(x) = 0$ . Without loss of generality, let  $x^* = 0$  be a fixed

point:  $j(x^*) = 0$ . Then in the neighbourhood of  $x^* = 0$  one has

$$j(x) = Bx + \frac{1}{2}x^T Gx + \dots, \quad \rho(x) = \rho(0) + \vec{q} \cdot x + \dots, \quad (26)$$

in which the Jacobian matrix  $B$  has elements  $B_{\ell k} = \partial j_\ell(0)/\partial x_k$ , tensor elements  $G_{\ell k h} = \partial^2 j_\ell(0)/\partial x_k \partial x_h$ , and gradient  $\vec{q}$  has elements  $q_k = \partial \rho(0)/\partial x_k$ . Then the Eq. (23) becomes

$$\rho(0)\text{Tr}[B] + \vec{q}Bx + \text{Tr}[B]\vec{q} \cdot x + \frac{\rho(0)}{2} \sum_{\ell k} \{G_{\ell \ell k} x_k + G_{\ell k \ell} x_k\} + O(\|x\|^2) = 0, \quad (27)$$

$\forall x$ . Since  $\rho(0) \neq 0$  this yields  $\text{Tr}[B] = 0$ . Furthermore,

$$\sum_{\ell} \frac{\partial \ln \rho(0)}{\partial x_\ell} \frac{\partial j_\ell(0)}{\partial x_k} + \frac{\partial^2 j_\ell(0)}{\partial x_\ell \partial x_k} = 0. \quad (28)$$

Hence, the fixed point of a canonical conservative system can not be a node or focus. It has to be either a saddle or a center, just as in a Hamiltonian system [29].

**Stationary points of  $\rho(x)$  in canonical conservative system.** Eq. (23) also indicates that  $\nabla \cdot j(x) = 0$  at the stationary position of  $\rho$  where  $\nabla \rho(x^s) = 0$ . A similar local analysis can be carried out near  $x^s$ , and one has

$$\sum_{\ell} j_\ell(x^s) \frac{\partial^2 \rho(x^s)}{\partial x_\ell \partial x_k} + \frac{\partial^2 j_\ell(x^s)}{\partial x_\ell \partial x_k} = 0. \quad (29)$$

**Planar canonical conservative system.** A planar canonical conservative system has the general form  $\dot{x} = \rho^{-1}(x, y) \partial H(x, y)/\partial y$ ,  $\dot{y} = -\rho^{-1}(x, y) \partial H(x, y)/\partial x$ . The phase portrait for this system is identical to the Hamiltonian system with  $\rho = 1$ . Indeed,  $H(x, y)$  is a conserved quantity in the dynamics:  $dH(x(t), y(t))/dt = 0$ .

**Mapping to microcanonical system via time change.** More generally, let  $\rho(x)j(x) = f(x)$ . Let  $\hat{x}(t)$  be a solution to the microcanonical conservative system  $\dot{x} = f(x)$ . Then the solution to the canonical conservative system with same initial value is  $x(t) = \hat{x}(\hat{t}(t))$  in which

$$\hat{t}(t) = t_0 + \int_{t_0}^t \rho^{-1}(\hat{x}(s)) ds.$$

## 5 General diffusion processes without detailed balance

We now bring the results from the above two sections to bear on the general diffusion processes.

**Decomposing free energy dissipation  $\frac{dF}{dt}$ .** In terms of  $j(x)$ , the “thermodynamic force” in a general diffusion generated by  $\mathcal{L}$  [28]

$$b(x) - A(x)\nabla \ln u(x) = j(x) - \left\{ A(x)\nabla \ln \left( \frac{u(x)}{\rho(x)} \right) \right\}, \quad (30)$$

in which the  $j(x)$  characterizes the deviation of  $b(x)$  from a gradient force; it does not involve  $u(x)$ . And since the  $\rho(x)$  is uniquely determined by the diffusion matrix and drift,  $j(x)$  is a non-dynamic term strictly determined by the  $A(x)$  and  $b(x)$ . The term in  $\{\dots\}$  characterizes non-stationarity of the  $u(x)$ . Then we have,

$$\left\langle \mathcal{L}(u), \rho \ln(\rho^{-1}u) \right\rangle = \left\langle \mathcal{L}_s(u), \rho \ln(\rho^{-1}u) \right\rangle + \left\langle \mathcal{L}_a(u), \rho \ln(\rho^{-1}u) \right\rangle. \quad (31)$$

The last term is zero for the canonical conservative system, e.g., Eq. (25). Therefore, the  $\frac{dF}{dt}$  for a general diffusion process with  $\mathcal{L}$  is entirely due to its symmetric part of the diffusion,  $\mathcal{L}_s$ . The canonical conservative dynamics generated by  $\mathcal{L}_a$  has no contribution toward the free energy dissipation of  $\mathcal{L}$ .

**Non-negative source for free energy  $F[u]$ .** Another important quantity from physics, the *house keeping heat* first proposed by Oono and Paniconi [35, 4], also called adiabatic entropy production [5], is

$$Q_{hk} = \int_{\mathbb{R}^n} (b - A\nabla \ln \rho) A^{-1}(x) (bu - A\nabla u) dx = \left\langle \rho j, A^{-1}(bu - A\nabla u) \right\rangle \quad (32)$$

for a general diffusion process. It is a type of projection of the thermodynamic driving force  $(b(x) - A(x)\nabla \ln u(x))$  onto the  $j(x)$ . It is zero for a reversible diffusion process with  $j(x) = 0$ . Noting the Eq. (30), we have

$$Q_{hk} = \left\langle \rho j, A^{-1}uj \right\rangle - \left\langle \rho j, \rho \nabla(\rho^{-1}u) \right\rangle = \left\langle \rho j, uA^{-1}j \right\rangle \geq 0. \quad (33)$$

Then the free energy dissipation for a general diffusion

$$\frac{dF(t)}{dt} = Q_{hk}(t) - e_p(t), \text{ or } e_p(t) = Q_{hk}(t) - \frac{dF(t)}{dt}. \quad (34)$$

All three terms  $e_p$ ,  $Q_{hk}$ ,  $-\frac{dF}{dt}$  are non-negative. A symmetric diffusion has  $Q_{hk}(t) = 0 \forall t$  (Eq. 14); a canonical conservative dynamics has  $\frac{dF(t)}{dt} = 0 \forall t$  (Eq. 25). A general diffusion process without detailed balance has  $e_p(t)$  which is consist of non-negative  $-\frac{dF}{dt}$  and  $Q_{hk}$ . More logically, we believe Eq. (34) should be interpreted as a balance equation for free energy  $F[u]$  with a source  $Q_{hk}$  and a sink  $e_p$ . The fact that both  $Q_{hk}$  and  $e_p$  are non-negative indicates that we have found the authentic source and sink terms for the free energy  $F[u]$ .



## 6 Ornstein-Uhlenbeck process: the linear case

We now consider the relationship between the above result and P. Ao's decomposition [22]. We shall explicitly work out the details for the Ornstein-Uhlenbeck (OU) Gaussian process [26, 21]. We show that in the linear case, the two theories are equivalent. The present approach is also equivalent to that of R. Graham's [20] under the assumption of large deviation principle.

We consider linear vector field  $b(x) = Bx$  and constant diffusion matrix  $A$ . The  $n$ -dimensional Fokker-Planck equation is

$$\frac{\partial u(x, t)}{\partial t} = \nabla (A \nabla u(x, t) - Bx u(x, t)). \quad (35)$$

It is easy to verify that the stationary solution has a Gaussian form [26]

$$\rho(x) = \frac{1}{(2\pi)^{n/2} \det \Xi} \exp \left\{ -\frac{1}{2} x^T \Xi^{-1} x \right\}, \quad (36)$$

in which the covariant matrix  $\Xi$  satisfies the Lyapunov matrix equation

$$B\Xi + \Xi B^T + 2A = 0. \quad (37)$$

Accordingly, we have

$$Bx = A \nabla \ln \rho(x) + j(x) = -A\Xi^{-1}x + j(x), \quad j(x) = Jx, \quad (38)$$

where matrix  $J = B + A\Xi^{-1}$ . We now show that matrix  $J$  can be written as  $-R\Xi^{-1}$  in which  $R$  is an anti-symmetric matrix. This is because of Eq. (37),

$$-R^T = (B\Xi + A)^T = \Xi B^T + A = -(B\Xi + A) = R.$$

Therefore,  $B = -(A + R)\Xi^{-1}$ . So the linear stochastic differential equation  $dx(t) = Bxdt + \Gamma dB(t)$ , with  $\frac{1}{2}\Gamma\Gamma^T = A$ , can be re-written as

$$\begin{aligned} Mdx(t) &= -\nabla \left( \frac{1}{2} x^T \Xi^{-1} x \right) dt + \Pi dB(t) \\ &= \nabla (\ln \rho(x)) dt + \Pi dB(t), \end{aligned} \quad (39)$$

where  $\Pi = (A + R)^{-1}\Gamma$  and  $M = (A + R)^{-1}$ . They are related via

$$M + M^T = \Pi\Pi^T, \quad (40)$$

because

$$\begin{aligned}\Pi^{-1} [(A + R)^{-1} + (A + R)^{-T}] \Pi^{-T} &= \Gamma^{-1} [(A - R) + (A + R)] \Gamma^{-T} \\ &= 2\Gamma^{-1} A \Gamma^{-T} = I.\end{aligned}$$

Eq. (39), together with (40), is Ao's form of stochastic differential equation [22]. For the linear system, one also has an additional property: the gradient field  $-\Xi^{-1}x$  and canonical conservative dynamics  $Jx$  are actually orthogonal:

$$(\Xi^{-1}x)^T \cdot (Jx) = -x^T \Xi^{-1} R \Xi^{-1} x = -(\Xi^{-1}x)^T R (\Xi^{-1}x) = 0. \quad (41)$$

This orthogonality was noted in both [20] and [21]. See [26] and [21] for more discussions on irreversible OU processes.

## 7 Discussion

The above result provides some insights into the structural stability of non-gradient vector field  $b(x)$ . Zeeman [36] has advocated the approach to structural stability based on  $\epsilon$ -noise perturbed dynamical systems. With respect to non-gradient field with a focus, he clearly noted one key difficulty in the Morse-Smale theory which requires a mapping from a focus to a node. While such a homeomorphism exists, "It is impossible, however, to make [such a map] smooth. ... Therefore in the attempt to capture density in two dimensions we have to abandon smoothness in the very definition of structural stability, and this, alas, is the beginning of the rot." [36]

For simplicity, let  $A(x) = \epsilon$ , then  $b(x) = j_\epsilon(x) + \epsilon \nabla \ln \rho_\epsilon(x)$  in which  $\rho_\epsilon(x)$  is the stationary density for the randomly perturbed dynamical system  $dx(t) = b(x)dt + \epsilon dB(t)$ , and  $j_\epsilon(x)$  is a canonical conservative system. In the limit of  $\epsilon \rightarrow 0$ , whether a limit exists for  $-\epsilon \ln \rho_\epsilon(x)$  is precisely the theory of large deviations [37, 38]. If a differentiable limit  $U(x)$  exists, then one obtains a decomposition  $b(x) = j_0(x) - \nabla U(x)$ . Moreover, since  $\nabla \cdot (\rho_\epsilon(x) j_\epsilon(x)) = 0 \forall \epsilon$ , one also has  $-\epsilon j_\epsilon(x) \cdot \nabla \ln \rho_\epsilon(x) = \epsilon \nabla \cdot j_\epsilon(x)$ . Thus  $j_0(x) \cdot \nabla U(x) = 0$  if the convergence is uniform, i.e., the decomposition is orthogonal. This is indeed R. Graham's theory [20, 38]. On the other hand, if  $\rho_\epsilon(x)$  has a nonzero limiting density  $\rho_0(x)$  on an attractor, then  $U(x) = 0$  on the entire attractor. This has been explicitly shown for systems with limit cycles and invariant tori where the asymptotic stationary density

$\rho_\epsilon(x) \sim \rho_0(x)e^{-U(x)/\epsilon}$  [11]. However, if the limit  $U(x)$  is non-differentiable or worse, some weaker forms might still exist [39]. This is the technical challenges encountered by R. Graham and his coworkers [20]; further investigations are required. The present work could provide a different approach to the challenge.

Canonical conservative system with  $\frac{dF(t)}{dt} = Q_{hk}(t) - e_p(t) = 0$  also leads to an interesting contradistinction between two views on time irreversibility. It is now known that entropy production  $e_p$  can be defined as the relative entropy between the probabilities of a trajectory  $\omega_t$  and its time reversal  $r(\omega_t)$ , under the probability measure generated by  $\mathcal{L}$  [3, 40]. For a stationary diffusion, the probability for the  $r(\omega_t)$  with the measure generated by  $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_a$  is in fact the same as the  $\omega_t$  under the measure generated by  $\mathcal{L}^- = \mathcal{L}_s - \mathcal{L}_a$  [40]. Now, if one introduces a different entropy production  $e_p^*$  as

$$e_p^* = E^{\mathbb{P}} \left[ \ln \left( \frac{d\mathbb{P}}{d\mathbb{P}^{-*}}(\omega_t) \right) \right], \quad (42)$$

in which the  $\mathbb{P}^{-*}$  is defined as the probability of time-reversed trajectory  $r(\omega_t)$  under the measure generated by  $\mathcal{L}^-$ . Then  $\mathbb{P}^{-*} = \mathbb{P}$  and  $e_p^* = 0$  for any stationary diffusion, as well as the canonical conservative dynamics. With such a choice for *the definition of time reversal*, there will be no nonequilibrium steady state; only equilibrium in which  $e_p^* = Q_{hk}^* = 0$ . Indeed, a Hamiltonian system is considered to be time reversible in classical dynamics precisely due to the second type of time reversal [41].

Finally, we note that the decomposition of a general diffusion process into an  $\mathcal{L}_a$  and an  $\mathcal{L}_s$  parts unifies nicely the earlier mathematical theories of dynamics formulated respectively by Newton and Fourier [42, 43].

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